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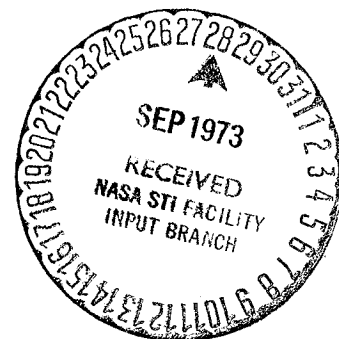
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PROPAGATION OF SOUND IN ELLIPTIC DUCTS

by

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SUMMARY AND CONCLUSIONS

The propagation of sound in elliptic ducts has been investigated as a model case to determine the acoustic effects of departures from symmetry in a circular duct.

The cut-off frequencies of the higher order circumferential modes in an elliptic duct have been calculated for various duct eccentricities. The lowest orders demonstrate an increase in cut-off frequency with eccentricity for the odd functions and a decrease for the even, in agreement with previous work. However the higher order modes show virtually no difference between odd and even functions and only a very small decrease in cut-off frequency compared with a circular duct of the same area. The practical implication of the results appears to be that even large deformations of the duct shape will have little effect providing duct area remains constant.

SOUND PROPAGATION IN ELLIPTIC DUCTS

Introduction

The propagation of sound in circular ducts is a phenomenon which has now been extensively studied because of its relevance to jet engine noise. The compressor or fan harmonics propagate down the duct as spinning modes. The spiral waves can couple into the rotating pressure patterns generated at the compressor face, and the behaviour of the spinning modes is found to dominate the characteristics of the harmonic components of jet noise radiation.

The circular duct problem was first studied, in an engine noise context, by Tyler and Sofrin¹. They showed how the very high harmonics (eigenfrequencies) of duct response could control the rotor noise levels observed, because of the ordered nature of the input pressure fluctuations from the blade/vane interaction process. In particular, for sufficiently low frequencies the radiation could be cut off, undergoing exponential decay within the duct. The cut-off conditions can be controlled by the choice of blade-vane numbers and have considerable attraction for engine noise control².

The overall noise radiation problem can be considered in two parts — first the propagation within the duct, and second the radiation from the end of the duct into free space. In principle these problems cannot be separated, but work by Morfey³ and others has shown how the interaction between the two parts of the radiation process is only significant for modes close to the cut-off condition. Most workers have assumed this separation of the propagation process to be acceptable and most have further modelled the radiation from the engine inlet via a baffled source model. Recently Lansing⁴ extended the work of Levine and Schwinger⁵ to include higher order radiation from an unbaffled duct more representative of an engine inlet, and has shown some degree of improvement between theory and experiment.

Thus at the present time the fundamental features of engine noise propagation in a circular duct are well established. Indeed Barry and Moore⁶ have shown how decay rates in a real fan inlet are remarkably close to the Tyler and Sofrin predictions. Several workers are now attempting to extend the theory to take account of flow within the duct, soft walls, and high intensity non-linear effects, and some success has been reported in each of these areas.

However a question which has not been studied in detail is the effect of departures from circular symmetry on the acoustic propagation. Hine⁷ investigated the effect of non-concentric annuli, and found little modification to the leading results. A further obvious question is the propagation in elliptic ducts. Undoubtedly several supposedly circular intakes are in fact mildly elliptic, and the acoustic effects of this have not, so far, been quantified. Furthermore some aircraft have adopted non-circular intakes for design convenience and there seems to be no reason in principle why elliptical intakes of quite large eccentricities could not be used on real aircraft. Thus propagation of sound in an elliptic duct appears to be a problem of some interest.

The equations governing acoustic propagation in an elliptic duct are well-known^{8 9 10}. They result from a separable solution to the wave equation in elliptic coordinates and have been investigated for several cases including membranes⁸ and electromagnetic wave guides¹⁰. However the studies have concentrated on the lowest order resonance frequency, so that little is known about the behaviour of the higher order modes in an elliptic duct. It is this problem which is of particular relevance to jet engine noise radiation. The results⁹ for the lowest order modes demonstrate a separation of the eigenfrequencies with the lowest order sine-elliptic result increasing in frequency and the cosine-elliptic decreasing compared to the circular duct case. This result would be of obvious interest from the noise control viewpoint if it occurred for the higher order modes.

In this report only propagation within an elliptic duct is studied. It is hoped that the radiation from an elliptic opening may be the subject of a later report. The solution to the problem is given in terms of Mathieu functions. Since these functions are not well-known, the derivation of the equations is given in some detail and for

comparative purposes. the equivalent problem for the circular duct is analysed first of all. A computer program to estimate the eigenfrequencies of the elliptic duct has been written, and results are presented in Section 4 of this report. A brief discussion of their practical implications for engine noise is given in Section 5.

2. Waves in a Circular Duct

Sound fields obey the wave equation which is

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad (1)$$

Transforming into cylindrical polar coordinates

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$t = t$$

gives

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$$

A separable solution $p = p_r(r) p_\theta(\theta) p_z(z) p_t(t)$ is then sought. Substituting into the above and dividing gives

$$\frac{1}{p_r} \frac{d^2 p_r}{dr^2} + \frac{1}{r p_r} \frac{dp_r}{dr} + \frac{1}{r^2 p_\theta} \frac{d^2 p_\theta}{d\theta^2} + \frac{1}{p_z} \frac{d^2 p_z}{dz^2} - \frac{1}{c^2 p_t} \frac{d^2 p_t}{dt^2} = 0$$

The z and t terms immediately separate out, equal to constants, taken as k_z^2 and $-\omega^2/c^2$ respectively, giving

$$\frac{d^2 p_z}{dz^2} = -k_z^2 p_z \quad \text{and} \quad \frac{d^2 p_t}{dt^2} = -\omega^2 p_t$$

and leaving

$$\frac{r^2}{p_r} \frac{d^2 p_r}{dr^2} + \frac{r}{p_r} \frac{dp_r}{dr} + r^2 (-k_z^2 + \frac{\omega^2}{c^2}) + \frac{1}{p_\theta} \frac{d^2 p_\theta}{d\theta^2} = 0$$

which is again separable, by putting

$$\frac{d^2 p_\theta}{d\theta^2} = -m^2 p_\theta \quad (2)$$

and so

$$\frac{d^2 p_r}{dr^2} + \frac{1}{r} \frac{dp_r}{dr} + (-k_z^2 + \frac{\omega^2}{c^2} - \frac{m^2}{r^2}) p_r = 0 \quad (3)$$

The solutions to the equations in t , z and θ can be conveniently taken as imaginary exponentials and the solution to the equation for p_r is of the form

$$p_r = A J_m(kr)$$

where

$$k^2 = \frac{\omega^2}{c^2} - k_z^2 \quad (4)$$

and J_m is a Bessel function of the first kind and m^{th} order. In fact the general solution of the equation to p_r also contains Bessel functions of the second kind, but these go to infinity at $r = 0$ and so must be rejected for the present duct propagation case where only finite pressures can be allowed along the axis of the duct.

Next the boundary conditions must be applied. For a hard wall duct these are that the normal velocity is zero at the duct walls $r = R$. The normal velocity at the wall is proportional to the radial derivative of the pressure so that this requires

$$\frac{\partial p}{\partial r} = 0 \quad \text{giving} \quad J_m'(kr) = 0$$

This is satisfied for a series of μ distinct values of k for each order m . These values will be denoted by $k_{m\mu}$, and are given in several reference works, e.g. Olver¹¹.

These values have a crucial effect on the sound radiation through equation (4), which may now be rewritten as

$$k_z = (\frac{\omega^2}{c^2} - k_{m\mu}^2)^{1/2}$$

Thus for any given value of $k_{m\mu}$ only values of ω which are greater than $ck_{m\mu}$ give rise to real, i.e. propagating, values of k_z . Smaller values of ω give rise to imaginary values of k_z which result in a decaying wave down the duct in the z -direction. This is the cut-off effect.

These decay rates can be very large and their significance in a practical situation has been demonstrated by several workers, e.g. Tyler and Sofrin¹, Smith², Morfey³, Barry and Moore⁶.

3. Waves in Elliptic Duct

In this case the wave equation (1) is transformed into elliptic coordinates using

$$\left. \begin{aligned} x &= h \cosh \xi \cos \eta \\ y &= h \sinh \xi \sin \eta \\ z &= z \\ t &= t \end{aligned} \right\} \quad (5)$$

This is a family of ellipses and hyperbolae as shown in Figure 1 with foci at $\pm h$. The η constant curves are hyperbolae. The ξ constant curves ellipses with major axis $2h \cosh \xi$ and minor axis $2h \sinh \xi$.

Under this transformation the wave equation becomes

$$\frac{2}{h^2(\cosh 2\xi - \cos 2\eta)} \left\{ \frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \eta^2} \right\} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$$

As before, a separable solution of the form

$$p = p_\xi(\xi) p_\eta(\eta) p_z(z) p_t(t)$$

is assumed and as before the z and t terms can be immediately removed as

$$\frac{d^2 p_z}{dz^2} = -k_z^2 p_z \quad \text{and} \quad \frac{d^2 p_t}{dt^2} = -\frac{\omega^2}{c^2} p_t$$

leaving

$$\frac{1}{p_\xi} \frac{d^2 p_\xi}{d\xi^2} + \frac{1}{p_\eta} \frac{d^2 p_\eta}{d\eta^2} + \left(\frac{\omega^2}{c^2} - k_z^2 \right) \frac{h^2}{2} (\cosh 2\xi - \cos 2\eta) = 0$$

which separates into

$$\frac{d^2 p_\eta}{d\eta^2} + \left\{ a - \left(\frac{\omega^2}{c^2} - k_z^2 \right) \frac{h^2}{2} \cos 2\eta \right\} p_\eta = 0 \quad (6)$$

$$\frac{d^2 p_\xi}{d\xi^2} + \left\{ \left(\frac{\omega^2}{c^2} - k_z^2 \right) \frac{h^2}{2} \cosh 2\xi - a \right\} p_\xi = 0 \quad (7)$$

'a' is a separation constant equivalent to m^2 term used in the circular case.

Equations (6) and (7) are versions of Mathieu equation which have the canonical forms, McLachlan⁸

$$\frac{d^2 p_\eta}{d\eta^2} + (a - 2q \cos 2\eta) p_\eta = 0 \quad (8)$$

and the modified Mathieu equation

$$\frac{d^2 p_\xi}{d\xi^2} - (a - 2q \cosh 2\xi)p_\xi = 0 \quad (9)$$

Thus in the present application

$$q = \left(\frac{\omega^2}{c^2} - k_z^2\right)\frac{h^2}{4} \quad (10)$$

and $4q/h^2$ plays the role of k in the circular case. However q is of additional significance here. As $q \rightarrow 0$, equation (8) tends to equation (2) already considered in the circular case. Indeed this would correspond to $h \rightarrow 0$ with the ellipses becoming circles. It can also be shown⁸ that as $q \rightarrow 0$ equation (9) tends to the Bessel equation (3).

The circular duct allows periodic sinusoidal waves of order m around its periphery, which could be of either sine or cosine form depending on their value at the origin assumed. In the elliptic duct case sine and cosine like waves can again occur but their origin is now related to the axis of symmetry of the ellipse. It is found that the odd "sine-elliptic" and even "cosine-elliptic" waves have distinctive features. The solutions to equation (8) are thus found in terms of the Mathieu functions

$$p_\eta = A_c ce_m(\eta, q) + A_s se_m(\eta, q) \quad (11)$$

where m is the order ($a \rightarrow m^2$ as $q \rightarrow 0$). Here we follow the notation of McLachlan⁸.

The solution of the modified Mathieu equation (9) is given in terms of the modified Mathieu functions of the first kind

$$p_\xi = B_c Ce_m(\xi, q) + B_s Se_m(\xi, q) \quad (12)$$

where Ce is the even "cosh-elliptic" function and Se is the odd "sinh-elliptic" function. In fact, as with the circular case, solutions of the second, third and fourth kind can also be found analogous to Hankel functions, but since these do not satisfy the continuity of pressure gradient on the interfocal line (see below), they must be rejected and will not be considered further here.

The conditions to be satisfied by the pressure function p in a reference plane $z = 0$ are as follows:

- (a) Since p is single-valued, it is periodic in η , with period at the most 2π

- (b) $p(\xi, \eta)$ is a continuous function, in particular, it is continuous across the interfocal line, so that

$$p(0, \eta) = p(0, -\eta)$$

- (c) also on crossing the interfocal line, we have the continuity of pressure gradient, so that

$$\frac{\partial p}{\partial \xi}(0, \eta) = -\frac{\partial p}{\partial \xi}(0, -\eta)$$

- (d) on the boundary, $\xi = \xi_0$, since the walls of the duct are supposed to be rigid, the component of pressure gradient normal to the wall (i.e. in the direction of ξ) vanishes at the walls, i.e. $\partial p / \partial \xi = 0$ at $\xi = \xi_0$.

Now because ce_m and se_m are periodic in η (and with period π or 2π , if m is a positive integer) they satisfy the condition (a).

If $Ce_m(\xi, q)$ is a solution of equation (9) we have $Ce_m(0, q)$ is a constant and also $ce_m(\eta) = ce_m(-\eta)$.

So $Ce_m(\xi)ce_m(\eta)$ satisfies condition (b).

Also since $Se_m(0, q) = 0$, $Se_m(0)se_m(\eta) = Se_m(0)se_m(-\eta)$ and so $Se_m(\xi)se_m(\eta)$ satisfies condition (b).

Also $se_m(\eta) = -se_m(-\eta)$, so that $Ce_m(\xi)se_m(\eta)$ would not satisfy (b).

It will be found that $Ce_m(\xi)ce_m(\eta)$ and $Se_m(\xi)se_m(\eta)$ combinations satisfy conditions (c) and are in fact the only acceptable combinations of Mathieu functions relevant to the problem.

Hence the permissible solution of the wave equation for the pressure field is given by

$$p = \left. \begin{matrix} C_m Ce_m(\xi, q) ce_m(\eta, q) \\ S_m Se_m(\xi, q) se_m(\eta, q) \end{matrix} \right\} e^{ik_z z} \cos(\omega t + \alpha) \quad (13)$$

which satisfy conditions (a), (b) and (c).

Now condition (d) requires that when $\xi = \xi_0$ (i.e. on the boundary of the elliptic duct) $\partial p / \partial \xi = 0$, which is

$$\left. \frac{d}{d\xi} Ce_m(\xi, q) \right|_{\xi=\xi_0} = 0 \quad \text{or} \quad \left. \frac{d}{d\xi} Se_m(\xi, q) \right|_{\xi=\xi_0} = 0 \quad (14)$$

These equations give (theoretically speaking), for a given value ξ_0 of ξ , an infinity of positive values of q , which satisfy the above equations, to each value of m . The first

equation gives a set of values $q_{m\mu}$ ($\mu = 0, 1, 2, \dots$) which satisfy it and the second equation gives a set of values $\bar{q}_{m\mu}$ (say) which satisfy it. These are known as parametric zeros of the functions. For each value of μ , both $q_{m\mu}$ and $\bar{q}_{m\mu}$, give nodal (zero pressure) ellipses in the reference plane. It may be noted that none of the $q_{m\mu}$ and $\bar{q}_{m\mu}$ are the same for the same value of m and μ .

4. Propagation and Decay of Waves in the Elliptic Duct

A fundamental relationship, from equation (10), is

$$k_z = \left(\frac{\omega^2}{c^2} - \frac{4q}{h^2} \right)^{1/2} \quad (15)$$

If the value $q_{m\mu}$ of q attained in Section 3 is such that $\omega/c < 2\sqrt{q_{m\mu}}/h$, an imaginary value of $k_{z\mu}$ results giving an exponential decay of waves along the axial direction of the duct.

If the value of $q_{m\mu}$ is such that $\omega/c \geq 2\sqrt{q_{m\mu}}/h$, real values of $k_{z\mu}$ occur so that this mode is propagated along the duct without attenuation.

For a particular m , q_{m0} gives the parametric zero such that if $\omega/c \geq 2\sqrt{q_{m0}}/h$, then the waves of this normal mode m are propagated; we can say that $2\sqrt{q_{m0}}/h$ gives the cut-off frequency for the particular mode m . The higher order radial modes associated with $q_{m\mu}$ are of less significance and will not be considered further here.

We thus require the values of these $2\sqrt{q_{m0}}/h$. The values obtained here are for the values of $m = 1(1)15$ and for ellipses of eccentricities $e = .1(.1)9$.

The results may be scaled for different size ellipses of the same eccentricity via an important well-known property connected with the reduced wave equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + k^2 p = 0 \quad (16)$$

for a given region R – see Kornhauser and Stakgold¹⁴.

If the above equation is valid under the boundary condition $\partial p / \partial n = 0$ on the boundary R , then there are infinity of values of k , k_i ($i = 0, 1, 2, \dots$),

$0 = k_0 < k_1 \leq k_2 \leq \dots$ which satisfy the above equation.

If A is the area of region R , and \bar{A} is the area of another region \bar{R} , similar to the

region R, and if \bar{k}_i are the values of k satisfying equation (16) for region \bar{R} , then

$$\frac{\bar{k}_i^2}{k_i^2} = \frac{A}{\bar{A}} \quad (17)$$

To obtain the values of the cut-off frequencies we shall consider a family of ellipses, each of which encloses the same area π . If the eccentricity of the ellipse is e, the ratio of the axes is $\sqrt{1 - e^2}$. The eigenvalues corresponding to any given elliptical boundary with the same eccentricity e can be found from equation (17).

$$\text{Now } 4q/h^2 = k^2.$$

$$\text{Since the area of the ellipse is } \pi, h^2 = e^2(1 - e^2)^{-1/2}$$

Let

$$s = 4q = k^2 h^2 = k^2 e^2 (1 - e^2)^{-1/2} \quad (18)$$

If $\xi = \xi_0$ on the boundary, we also have

$$\cosh \xi_0 = \frac{1}{e}$$

If we write $u = \sqrt{q} \exp \xi_0$ and $v = \sqrt{q} \exp(-\xi_0)$ we have

$$w_0 = u + v = 2\sqrt{q} \cosh \xi_0 = \sqrt{s} \cosh \xi_0$$

therefore

$$w_0^2 = \frac{s}{e^2} = k^2 (1 - e^2)^{-1/2} \quad (19)$$

It is not possible in practice to determine the parametric zeros by a direct evaluation. The procedure adopted here was as follows:

The radial (modified) Mathieu functions can be expanded in a rapidly converging Bessel function series with argument $\sqrt{s} \cosh \xi$. The coefficients of these series for given values of s are tabulated in Reference 13. Taking a particular value of m, w_0 is calculated for each value of s. The computer program evaluates the values proportional to Ce'_m and Se'_m for incremental values of $\sqrt{s} \cosh \xi$ starting from \sqrt{s} at an interval of 0.5 (for orders $m = 1, 2$, an increment of 0.1 has been given) and when the function changes sign from positive to negative the zero is found by means of iteration. The accuracy of the program was checked to eight significant figures for lower values of m and up to six significant figures for a higher m. Once the value of e is obtained the value of k can be evaluated with the help of the equation 19.

Though these are the values that are required for the cut-off frequencies, the values of e obtained are not those desired, and are at unequal intervals. However, the required values of the cut-off frequencies for $e = .1(.1).9$ can be found by interpolation. These values are tabulated in Tables I and II for even and odd radial functions respectively for orders $m = 1 - 15$. These tables have an error of maximum of 3 units in the third decimal place.

Discussion

The remarkable feature of the results of Tables I and II is their small deviation from the circular case. The results of Daymond⁹ for the lowest order modes demonstrated an increase in cut-off frequency for the odd functions and a decrease for the even, and the computations presented here were undertaken in the expectation that this effect would carry over to the higher modes. However, while the present results are entirely consistent with Daymond for the lower modes they also demonstrate that the higher order odd and even modes are virtually indistinguishable, with only a very small reduction in cut-off frequency from the circular case.

A feature of the present results is the reference of all calculations to ellipses of equal area. It has been usual to take other scale references — for instance the major axis or interfocal distance — and this would mask the effect noted here. Attempts have been made to predict this effect analytically. These have not met with success, although it would appear that a very strongly convergent solution should exist based on the Bessel function zeros.

The practical implication of these results is that even major deformations of inlet shape which retain equal inlet area will have virtually no influence on the cut-off conditions of the sound radiated. How far these elliptical results could be carried over to other shapes is, of course, questionable. Nevertheless, this result would appear to give a useful insight for the interpretation of engine noise data.

The possible use of elliptic intakes as a noise control device seems to be unrewarding, at least from the point of view of cut-off frequency. However two other effects could perhaps be of value here. Firstly the attenuation of any absorbent liners would probably

increase as the duct walls come closer together. Secondly there could be possible advantages to be gained from the directionality pattern of the sound radiated from an elliptic duct, which could allow some azimuthal redistribution of the noise field.

Table I -- Even Functions

$\begin{matrix} e \\ m= \end{matrix}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.8412	1.838	1.825	1.803	1.768	1.720	1.670	1.575	1.442	1.235
2	3.0542	3.054	3.053	3.048	3.030	3.000	2.932	2.815	2.620	2.273
3	4.2012	4.200	4.200	4.197	4.190	4.168	4.115	4.000	3.762	
4	5.3176	5.317	5.317	5.315	5.307	5.287	5.240	5.130	4.872	4.277
5	6.4156	6.415	6.415	6.413	6.405	6.382	6.330	6.228	5.958	5.278
6	7.5013	7.500	7.500	7.498	7.488	7.465	7.413	7.300	7.017	6.252
7	8.5778	8.578	8.578	8.572	8.562	8.537	8.482	8.357	8.062	7.225
8	9.6474	9.647	9.645	9.643	9.630	9.603	9.580	9.408	9.093	
9	10.7114	10.710	10.710	10.705	10.692	10.662	10.585	10.453		
10	11.7709	11.770	11.768	11.765	11.753	11.720	11.645	11.487		
11	12.8265	12.285	12.822	12.820	12.805	12.770	12.690			
12	13.8788	13.875	13.873	13.870	13.855	13.820	13.738			
13	14.9284	14.928	14.928	14.923	14.905	14.862	14.775			
14	15.9754	15.975	15.975	15.968	15.947	15.905				
15	17.0203	17.020	17.017	17.010	16.992	16.948				

Table II -- Odd Functions

$\begin{matrix} e \\ m= \end{matrix}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1	1.8412	1.855	1.877	1.890	1.920	1.960	2.027	2.138	2.312	2.650
2	3.0542	3.055	3.058	3.060	3.062	3.068	3.083	3.110	3.180	3.380
3	4.2012	4.185	4.172	4.188	4.193	4.182	4.165	4.140	4.118	4.152
4	5.3176	5.317	5.317	5.315	5.307	5.290	5.257	5.195	5.095	4.973
5	6.4156	6.145	6.410	6.402	6.382	6.370	6.252	6.093	5.822	
6	7.5013	7.500	7.500	7.498	7.488	7.463	7.412	7.310	7.097	6.695
7	8.5778	8.578	8.578	8.572	8.562	8.565	8.480	8.360	8.110	
8	9.6474	9.647	9.645	9.643	9.630	9.625	9.540	9.410	9.120	
9	10.7114	10.710	10.710	10.705	10.693	10.662	10.595	10.450		
10	11.7709	11.770	11.767	11.762	11.750	11.717	11.645	11.485		
11	12.8265	12.825	12.823	12.820	12.805	12.770	12.690			
12	13.8788	13.875	13.872	13.870	13.855	13.823	13.732			
13	14.9284	14.928	14.928	14.922	14.905	14.862	14.775			
14	15.9754	15.975	15.975	15.967	15.947	15.905				
15	17.0203	17.020	17.018	17.010	16.993	16.948				

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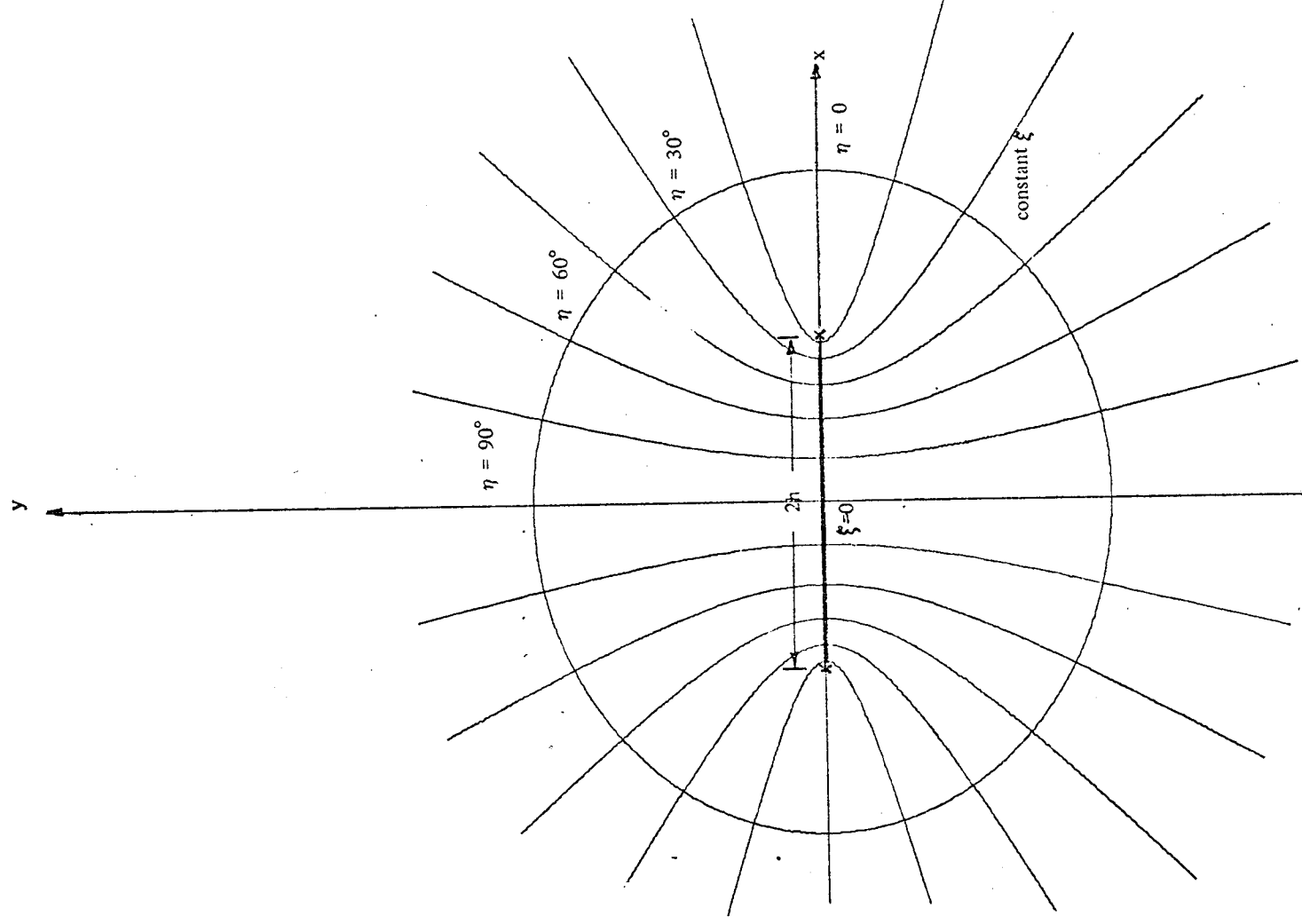


Figure 1: Coordinate System for Elliptic Duct